

ON THE LUCAS PROPERTY OF LINEAR RECURRENT SEQUENCES

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ABSTRACT. Let S be an arithmetic function. S has Lucas property if for any prime p and $n = \sum_{i=0}^r n_i p^i$, where $0 \leq n_i \leq p-1$,

$$S(n) \equiv S(n_0)S(n_1) \dots S(n_r) \pmod{p}. \quad (0.1)$$

In this note, we discuss the Lucas property of Fibonacci sequences and Lucas numbers. Meanwhile, we find some other interesting results.

1. INTRODUCTION

The famous Lucas' theorem states that

$$\binom{n}{m} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \dots \binom{n_r}{m_r} \pmod{p}, \quad (1.1)$$

where $n, m \in \mathbb{N}$, the base p expansions of n and m are $n = \sum_{i=0}^r n_i p^i$, $m = \sum_{i=0}^r m_i p^i$ ($0 \leq n_i, m_i \leq p-1$).

In 1992, Richard J. McIntosh [5] gave a definition of the Lucas property and the double Lucas property, i.e.,

Definition 1.1. Let S be an arithmetic function. S has Lucas property if for any prime p and $n = \sum_{i=0}^r n_i p^i$, where $0 \leq n_i \leq p-1$,

$$S(n) \equiv S(n_0)S(n_1) \dots S(n_r) \pmod{p}. \quad (1.2)$$

And let D be a bivariate arithmetic function. D has double Lucas property if for any prime p , $n = \sum_{i=0}^r n_i p^i$, and $m = \sum_{i=0}^r m_i p^i$, where $0 \leq n_i, m_i \leq p-1$,

$$D(n, m) \equiv D(n_0, m_0)D(n_1, m_1) \dots D(n_r, m_r) \pmod{p}. \quad (1.3)$$

Another way of stating this is to say that S is an LP function and D is a DLP function.

There are numerous examples: a^n is an LP function for any rational number a ; the Apéry numbers $A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ is an LP function (Cf. Gessel [4]); the function $\omega(n)$ defined by

$$\frac{1}{J_0(2z^{1/2})} = \sum_{n=0}^{\infty} \omega(n) \frac{z^n}{(n!)^2}$$

is an LP function (Cf. Carlitz [2]); and according to Lucas' theorem, the binomial coefficient $D(n, m) = \binom{n}{m}$ is a DLP function.

Moreover, we add another definition.

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Definition 1.2. Let S be an arithmetic function. S has Lucas property with the prime p if for any $n = \sum_{i=0}^r n_i p^i$, where $0 \leq n_i \leq p-1$,

$$S(n) \equiv S(n_0)S(n_1) \dots S(n_r) \pmod{p}. \quad (1.4)$$

It can be said that S is an LP function with the prime p .

In this paper, we discuss the Lucas property of Fibonacci and Lucas numbers.

Let F_n be the Fibonacci sequence, i.e., $F_n : F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \ (n \geq 2)$, and L_n be the Lucas numbers $L_n : L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2} \ (n \geq 2)$.

We obtain the following theorems.

Theorem 1.1. *Let a, b be two positive integers. Then $S(n) = F_{an+b}$ is an LP function with the prime p if and only if*

$$\begin{cases} F_a & \equiv 0 \pmod{p}, \\ F_b & \equiv 1 \pmod{p}. \end{cases} \quad (1.5)$$

For Lucas numbers, we have

Theorem 1.2. *Let a, b be two positive integers. Then $S(n) = L_{an+b}$ is an LP function with the prime p if and only if*

$$\begin{cases} 5F_a & \equiv 0 \pmod{p}, \\ F_b & \equiv 1 \pmod{p}. \end{cases} \quad (1.6)$$

From these two theorems, we can obtain some corollaries.

Corollary 1.1. *Let a and b be positive integers. Then $S(n) = F_{an+b}$ is not an LP function and L_{an+b} is not an LP function.*

Proof. The proof is by contradiction. Let a and b be positive integers such that $S(n) = F_{an+b}$ is an LP function. Then by Theorem 1.1, p divides F_a for any prime p , a contradiction. A similar proof follows for $S(n) = L_{an+b}$. \square

Corollary 1.2. *Let $p = 5$. Then for any positive integer a ,*

(1) $S(n) = F_{5an+b}$ is an LP function with the prime 5, where $b \equiv 1, 2, 8$ or $19 \pmod{20}$.

(2) $S(n) = L_{an+b}$ is an LP function with the prime 5, where $b \equiv 1 \pmod{4}$.

Corollary 1.3. *Let p be a Fibonacci prime, namely, there exists a positive integer a such that $F_a = p$. Then F_{an+1} is an LP function with the prime p and L_{an+1} is an LP function with the prime p .*

More generally, let $\alpha(p) := \min\{n \mid p \text{ divides } F_n\}$ for a prime p . Then we have

Corollary 1.4. *The condition $F_a \equiv 0 \pmod{p}$ in Theorem 1.1 can be replaced by $a = \alpha(p)k$, where k is an arbitrary positive integer. And if $p \neq 5$, the condition $5F_a \equiv 0 \pmod{p}$ in Theorem 1.2 can also be replaced by $a = \alpha(p)k$, where k is an arbitrary positive integer.*

Proof. For any integers m, n , $\gcd(F_m, F_n) = F_{\gcd(m, n)}$. Hence, $\gcd(F_a, F_{\alpha(p)}) = F_{\gcd(a, \alpha(p))}$. And if $F_a \equiv 0 \pmod{p}$, then $p | F_{\gcd(a, \alpha(p))}$. From the definition of $\alpha(p)$, we obtain that $\gcd(a, \alpha(p)) = \alpha(p)$. So, $a = \alpha(p)k$ for some integer k .

Similarly, for any positive integer k , $\gcd(F_{\alpha(p)k}, F_{\alpha(p)}) = F_{\gcd(\alpha(p)k, \alpha(p))} = F_{\alpha(p)}$. Hence, $F_{\alpha(p)k} \equiv 0 \pmod{p}$. \square

A natural extension of these two theorems is to look at the Lucas property of general linear recurrent sequences. We obtain an analogous result to the two theorems above.

Theorem 1.3. *Let A_n be a linear recurrent sequence, i.e., $\{A(n)\}$ satisfies the linear recurrent relation:*

$$A_n = uA_{n-1} + vA_{n-2} \quad (n \geq 2),$$

where A_0, A_1, u and v are all integers. Then for any integers a and b , $S(n) = A_{an+b}$ is an LP function with the prime p if and only if

$$\begin{cases} vs(a-1, u, v)(vA_0^2 + uA_0A_1 - A_1^2) & \equiv 0 \pmod{p}, \\ A_b & \equiv 1 \pmod{p}. \end{cases} \quad (1.7)$$

where

$$s(k, u, v) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} u^{k-2i} v^i.$$

When it comes to the generalizations of Fibonacci numbers, we obtain two more corollaries.

Corollary 1.5. *Let $\{A_n\}$ be an Lucas sequence or $(P, -Q)$ -Fibonacci sequence, that is, $A_0 = 0, A_1 = 1, u = P$, and $v = -Q$. Then for any integers a and b , $S(n) = A_{an+b}$ is an LP function with the prime p if and only if*

$$\begin{cases} Qs(a-1, P, -Q)A_1^2 & \equiv 0 \pmod{p}, \\ A_b & \equiv 1 \pmod{p}. \end{cases} \quad (1.8)$$

In particular, when $\{A_n\}$ are Pell numbers, $S(n) = A_{an+b}$ is an LP function with the prime p if and only if

$$\begin{cases} s(a-1, 2, 1) & \equiv 0 \pmod{p}, \\ A_b & \equiv 1 \pmod{p}. \end{cases} \quad (1.9)$$

Another famous generalization of Fibonacci numbers is Fibonacci word, which is in the case of $u = v = 1$. Similarly, we have

Corollary 1.6. *Let $\{A_n\}$ be Fibonacci words. Then for any integers a and b , $S(n) = A_{an+b}$ is an LP function with the prime p if and only if*

$$\begin{cases} F_a(A_0^2 + A_0A_1 - A_1^2) & \equiv 0 \pmod{p}, \\ A_b & \equiv 1 \pmod{p}. \end{cases} \quad (1.10)$$

where F_a is the a th Fibonacci number.

2. PRELIMINARIES

For a fixed prime p , the following two corollaries from McIntosh [5] will be needed.

Lemma 2.1. *Let $S(n)$ be an LP function with the prime p , which is not identically zero. Then $S(0) \equiv 1 \pmod{p}$.*

Lemma 2.2. *$S(n)$ is an LP function with the prime p , and $S(n)$ is periodic modulo p if and only if $S(n) \equiv S(1)^n \pmod{p}$.*

Meanwhile, we can get the following lemma by induction on n .

Lemma 2.3. *Let n be a positive integer. Then*

$$(1) \quad F_n \equiv n3^{n-1} \pmod{5}. \quad (2.1)$$

$$(2) \quad L_n \equiv 3^{n-1} \pmod{5}. \quad (2.2)$$

Remarks. By using Lemma 2.2 and Lemma 2.3, we can find some LP functions with the prime 5,

- (1) $S(n) = F_{5n+b}$ is an LP function with the prime 5, where $b \equiv 1, 2, 8$ or $19 \pmod{20}$.
- (2) $S(n) = L_{n+1}$ is an LP function with the prime 5.

In order to get the theorems, we need one more lemma.

Lemma 2.4. *Let n, r be two integers. Then*

$$(1) \text{ (Catalan's identity)} \quad F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r} \cdot F_r^2. \quad (2.3)$$

$$(2) \quad L_{n+r}L_{n-r} - L_n^2 = (-1)^{n-r} \cdot 5F_r^2. \quad (2.4)$$

$$(3) \quad A_{n+r}A_{n-r} - A_n^2 = (-v)^{n-r} s^2(r-1, u, v)(vA_0^2 + uA_0A_1 - A_1^2). \quad (2.5)$$

Proof of (2.4). We prove it by using the determinant of the matrix and the fact that

$$\begin{cases} L_{n+r} &= F_{r+1}L_n + F_rL_{n-1}, \\ L_n &= F_{r+1}L_{n-r} + F_rL_{n-r-1}. \end{cases}$$

Hence,

$$\begin{aligned} L_{n+r}L_{n-r} - L_n^2 &= \begin{vmatrix} L_{n+r} & L_n \\ L_n & L_{n-r} \end{vmatrix} \\ &= \begin{vmatrix} F_{r+1}L_n + F_rL_{n-1} & L_n \\ F_{r+1}L_{n-r} + F_rL_{n-r-1} & L_{n-r} \end{vmatrix} \\ &= F_r \begin{vmatrix} L_{n-1} & L_n \\ L_{n-r-1} & L_{n-r} \end{vmatrix} \\ &= F_r \begin{vmatrix} L_{n-1} & L_{n-2} \\ L_{n-r-1} & L_{n-r-2} \end{vmatrix} \\ &= \dots \end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-r} F_r \begin{vmatrix} L_{r+1} & L_r \\ L_1 & L_0 \end{vmatrix} \\
&= (-1)^{n-r} F_r (2L_{r+1} - L_r) \\
&= (-1)^{n-r} F_r (L_{r+1} + L_{r-1}) \\
&= (-1)^{n-r} \cdot 5F_r^2.
\end{aligned}$$

So, (2.4) is true. \square

Proof of (2.5). To prove (2.5), we first prove that

$$A_{n+r} = s(k, u, v)A_{n+r-k} + t(k, u, v)A_{n+r-k-1}, \quad (2.6)$$

where $s(k, u, v) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} u^{k-2i} v^i$ and $t(k, u, v) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-j}{j} u^{k-1-2j} v^{j+1}$. For $k=1$, (2.6) holds. By inducting on k , we can obtain the result. Assume for $k = 1, 2, \dots, m$, (2.6) holds. For $k = m+1$,

$$\begin{aligned}
A_{n+r} &= s(m, u, v)A_{n+r-m} + t(m, u, v)A_{n+r-m-1} \\
&= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m-2i} v^i A_{n+r-m} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} A_{n+r-m-1} \\
&= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m-2i} v^i (uA_{n+r-m-1} + vA_{n+r-m-2}) + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} A_{n+r-m-1} \\
&= \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m+1-2i} v^i + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} \right) A_{n+r-m-1} \\
&\quad + \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m-2i} v^{i+1} A_{n+r-m-2} \\
&= \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m+1-2i} v^i + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} \right) A_{n+r-m-1} + t(m+1, u, v)A_{n+r-m-2}.
\end{aligned}$$

If $m \equiv 0 \pmod{2}$,

$$\begin{aligned}
&\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m+1-2i} v^i + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} \\
&= \sum_{i=0}^{\frac{m}{2}} \binom{m-i}{i} u^{m+1-2i} v^i + \sum_{j=0}^{\frac{m}{2}-1} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} \\
&= u^{m+1} + \sum_{i=1}^{\frac{m}{2}} \left(\binom{m-i}{i} + \binom{m-i}{i-1} \right) u^{m+1-2i} v^i
\end{aligned}$$

$$\begin{aligned}
&= u^{m+1} + \sum_{i=1}^{\frac{m}{2}} \binom{m+1-i}{i} u^{m+1-2i} v^i \\
&= s(m+1, u, v).
\end{aligned}$$

If $m \equiv 1 \pmod{2}$,

$$\begin{aligned}
&\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} u^{m+1-2i} v^i + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} \\
&= \sum_{i=0}^{\frac{m-1}{2}} \binom{m-i}{i} u^{m+1-2i} v^i + \sum_{j=0}^{\frac{m-1}{2}} \binom{m-1-j}{j} u^{m-1-2j} v^{j+1} \\
&= v^{\frac{m+1}{2}} + \sum_{i=0}^{\frac{m-1}{2}} \left(\binom{m-i}{i} + \binom{m-i}{i-1} \right) u^{m+1-2i} v^i \\
&= v^{\frac{m+1}{2}} + \sum_{i=0}^{\frac{m-1}{2}} \binom{m+1-i}{i} u^{m+1-2i} v^i \\
&= s(m+1, u, v).
\end{aligned}$$

Hence, $A_{n+r} = s(m+1, u, v)A_{n+r-m-1} + t(m+1, u, v)A_{n+r-m-2}$, which means (2.6) holds. The rest of the proof is similar to the proof of (2.4). By using the determinant of the matrix, we can obtain

$$A_{n+r}A_{n-r} - A_n^2 = (-1)^{n-r} v^{n-r-1} t(r, u, v) (A_{r+1}A_0 - A_rA_1).$$

By using (2.6) and the fact $t(r, u, v) = vs(r-1, u, v)$, we have

$$\begin{aligned}
A_{n+r}A_{n-r} - A_n^2 &= (-1)^{n-r} v^{n-r-1} t(r, u, v) (A_{r+1}A_0 - A_rA_1) \\
&= (-1)^{n-r} v^{n-r-1} t(r, u, v) (t(r, u, v)A_0^2 - s(r-1, u, v)A_1^2 + s(r, u, v)A_1A_0 - t(r-1, u, v)A_0A_1) \\
&= (-v)^{n-r} s(r-1, u, v) (vs(r-1, u, v)A_0^2 - s(r-1, u, v)A_1^2 + s(r, u, v)A_1A_0 - vs(r-2, u, v)A_0A_1) \\
&= (-v)^{n-r} s^2(r-1, u, v) (vA_0^2 + uA_0A_1 - A_1^2).
\end{aligned}$$

So, (2.5) is true. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. The Fibonacci numbers are periodic modulo p for any prime p . So is $S(n) = F_{an+b}$, where a, b are positive integers.

We first prove the necessity. Assume that $S(n) = F_{an+b}$ is an LP function with the prime p . From Lemma 2.1, $S(0) \equiv 1 \pmod{p}$, so $F_b \equiv 1 \pmod{p}$. And from Lemma 2.2, for any positive integer n , $F_{an+b} \equiv F_{a+b}^n \pmod{p}$. Set $n = 2$, $F_{2a+b} \equiv F_{a+b}^2 \pmod{p}$. By using Catalan's identity (2.3), we have

$$\begin{aligned}
F_{a+b+a}F_{a+b-a} &= F_{a+b}^2 - (-1)^{a+b-a} F_a^2 \\
F_{2a+b}F_b &= F_{a+b}^2 - (-1)^b F_a^2
\end{aligned}$$

$$\begin{aligned}
F_{a+b}^2 &\equiv F_{a+b}^2 - (-1)^b F_a^2 \pmod{p} \\
F_a^2 &\equiv 0 \pmod{p} \\
F_a &\equiv 0 \pmod{p}.
\end{aligned}$$

Hence, a and b satisfy

$$\begin{cases} F_a &\equiv 0 \pmod{p} \\ F_b &\equiv 1 \pmod{p} \end{cases}$$

Next we prove the sufficiency. From Lemma 2.2, we have to prove that

$$S(n) \equiv S(1)^n \pmod{p}. \quad (3.1)$$

And we'll prove it by induction on n . For $n = 1$, it's obviously true. Assume that for $n \leq k$, (3.1) holds. For $n = k + 1$, by using Catalan's identity (2.3), we have

$$\begin{aligned}
F_{ak+b+a}F_{ak+b-a} &= F_{ak+b}^2 - (-1)^{ak+b-a} F_a^2 \\
F_{a(k+1)+b}F_{a(k-1)+b} &= F_{ak+b}^2 - (-1)^{a(k-1)+b} F_a^2 \\
F_{a(k+1)+b}F_{a+b}^{k-1} &\equiv F_{a+b}^{2k} - (-1)^{a(k-1)+b} F_a^2 \pmod{p} \\
F_{a(k+1)+b} &\equiv F_{a+b}^{k+1} \pmod{p}.
\end{aligned}$$

Hence, (3.1) holds for any positive integer n . And F_{an+b} is an LP function with the prime p . \square

Proof of Theorem 1.2. The proof is similar to Theorem 1.1. Lucas number is periodic modulo p for any prime p . So is $S(n) = L_{an+b}$, where a, b are positive integers. We first prove the necessity. Assume that $S(n) = L_{an+b}$ is an LP function with the prime p . From Lemma 2.1, $S(0) \equiv 1 \pmod{p}$, so $L_b \equiv 1 \pmod{p}$. And from Lemma 2.2, for any positive integer n , $L_{an+b} \equiv L_{a+b}^n \pmod{p}$. Set $n = 2$, $L_{2a+b} \equiv L_{a+b}^2 \pmod{p}$. By using (2.4), we have

$$\begin{aligned}
L_{a+b+a}L_{a+b-a} &= L_{a+b}^2 + (-1)^{a+b-a} \cdot 5F_a^2 \\
L_{2a+b}L_b &= L_{a+b}^2 + (-1)^b \cdot 5F_a^2 \\
L_{a+b}^2 &\equiv L_{a+b}^2 + (-1)^b \cdot 5F_a^2 \pmod{p} \\
5F_a^2 &\equiv 0 \pmod{p} \\
5F_a &\equiv 0 \pmod{p}.
\end{aligned}$$

Hence, a and b satisfy

$$\begin{cases} 5F_a &\equiv 0 \pmod{p} \\ L_b &\equiv 1 \pmod{p} \end{cases}$$

Next we prove the sufficiency. From Lemma 2.2, we also have to prove that (3.1) is true. And we'll prove it by induction on n . For $n = 1$, it's obviously true. Assume that for $n \leq k$, (3.1) holds. For $n = k + 1$, by using (2.4), we have

$$\begin{aligned}
L_{ak+b+a}L_{ak+b-a} &= L_{ak+b}^2 + (-1)^{ak+b-a} \cdot 5F_a^2 \\
L_{a(k+1)+b}L_{a(k-1)+b} &= L_{ak+b}^2 + (-1)^{a(k-1)+b} \cdot 5F_a^2 \\
L_{a(k+1)+b}L_{a+b}^{k-1} &\equiv L_{a+b}^{2k} + (-1)^{a(k-1)+b} \cdot 5F_a^2 \pmod{p} \\
L_{a(k+1)+b} &\equiv L_{a+b}^{k+1} \pmod{p}.
\end{aligned}$$

Hence, (3.1) holds for any positive integer n . And L_{an+b} is an LP function with the prime p . \square

Proof of Theorem 1.3. From [3] and [6], we know that for any integer m , a linear recurrent sequence of integers modulo m is periodic. The same is true for a prime p . Hence $S(n) = A_{an+b}$ is periodic modulo p . To obtain the proof it is enough to apply the reasoning just like in the proofs of Theorem 1.1 and Theorem 1.2. \square

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